Mechanics and Physics of Porous Solids

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Based on a lecture from O. Coussy and M. Vandamme

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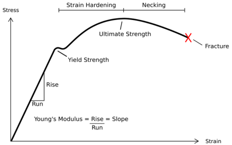
Definition

What is mechanics ?

Mechanics is the study of the movement and deformation of physical systems.

A porous solid consists in a solid matrix filled by one or more fluid (gas or liquid). To consider the mechanics of a porous solid we need to know the *mechanics of deformable solids* and *the mechanics of fluids* and the coupling between both.



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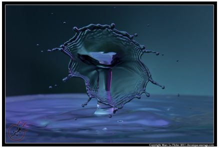

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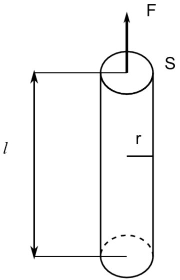
Simple deformations, elasticity

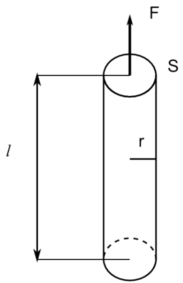


Cylinder subjected to a force F

The strain (deformation) of the cylinder is defined as: $\varepsilon = \frac{\Delta l}{l_0}$

The stress applying on the cylinder is defined as $\sigma = \frac{F}{S}$

Simple deformations, elasticity



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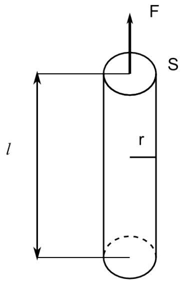
Hooke's law: *ut tensio sic vis*

If the stress is moderate, the cylinder is within the elastic domain:

- the deformation is linear with the stress $\sigma = E\varepsilon$, E is called Young Modulus ($\sim GPa$)
- the deformation is reversible

Simple deformations, elasticity



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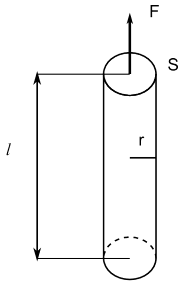

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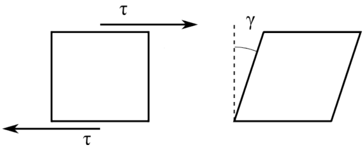
Poisson, bulk moduli

If the stress is moderate, the cylinder is within the elastic domain:

- the change of radius is characterized by the Poisson modulus: $\frac{\Delta r}{r_0} = -\nu \cdot \varepsilon$
- for an isotropic deformation (pressure), the volume change is: $\Delta P = -K \frac{\Delta V}{V}$

Simple deformations, elasticity

Shear



If the force is parallel to the surfaces it is called shear

The variation of the angle γ (shear angle) is also linear under the elasticity assumption

$\tau = G\gamma$, with G the shear modulus

Relation between the different moduli

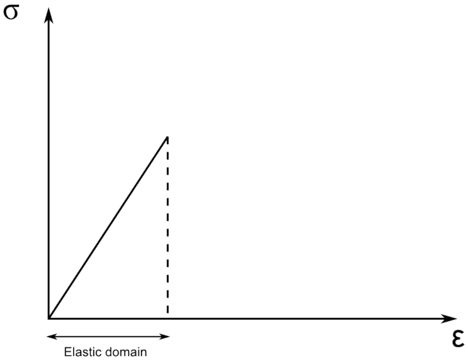
- $G = \frac{E}{2(1+\nu)}$
- $\frac{1}{E} = \frac{1}{9K} + \frac{1}{3G}$
- $K = \frac{1}{3}\frac{E}{1-2\nu}$
- if $\nu > 0.5$ the volume increases under isotropic compression $\Rightarrow$ Non physical
  if $\nu < 0.5$ the volume decreases under compression and increases under traction

Usual values of the mechanical moduli for different materials

Material	E (GPa)	G (GPa)	ν	K (GPa)
Rubber	0.001 to 0.1	0.0003	0.48 to 0.5	0.0001 to 0.1
Steel	210	77	0.3	160
Sandstone	3 to 90	1 to 40	0.2 to 0.35	2 to 110
Limestone	9 to 80	3 to 30	0.2 to 0.3	5 to 70
Granite	10 to 70	4 to 30	0.1 to 0.2	50
Concrete	20 to 50	21	0.1 to 0.3	15
Cork	0.0186	0.0093	0	0.0062

Beyond elasticity, plasticity and rupture

Response of a system under a simple traction test



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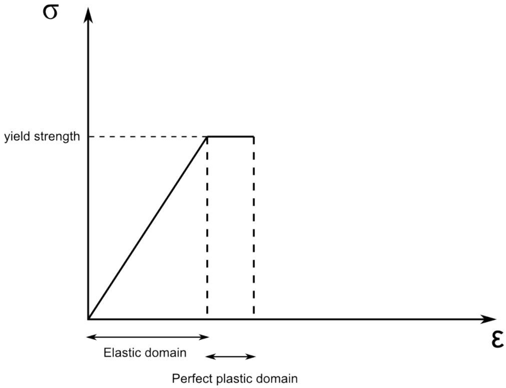

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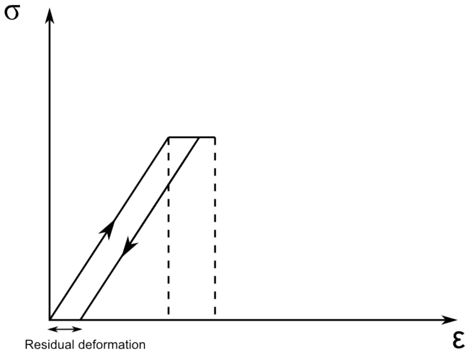

Beyond elasticity, plasticity and rupture

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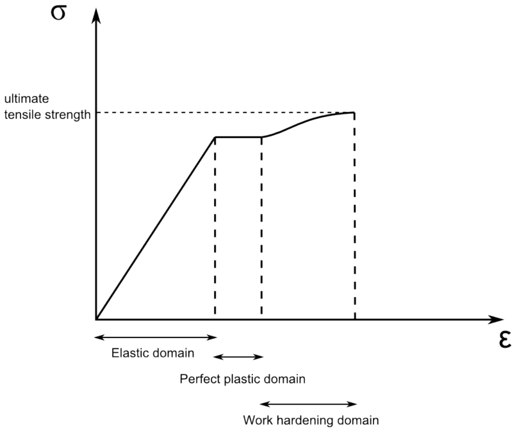

Beyond elasticity, plasticity and rupture

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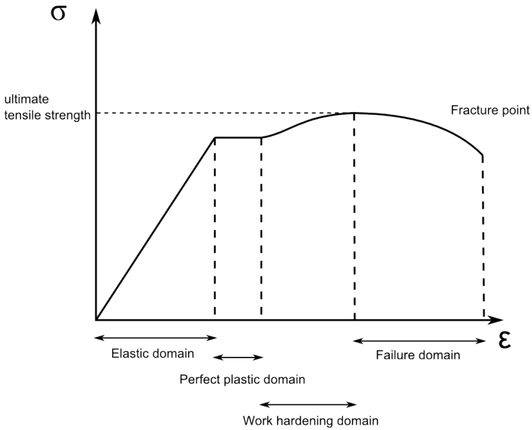
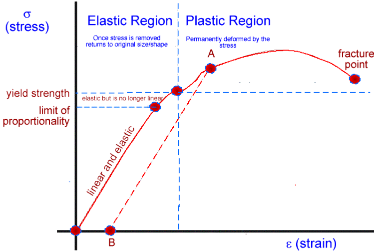


Beyond elasticity, plasticity and rupture

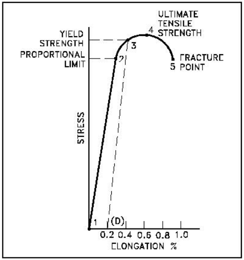
Brittle vs ductile

When the stresses are too large, the system response is not linear anymore

- if the system is brittle, it will break when the stress/strain will reach a critical value
- if the system is ductile, it will sustain an irreversible deformation before finally breaking



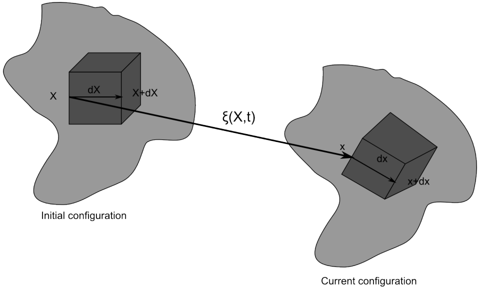
ductile fracture



brittle material

Strains and strain tensor

How to describe more complex solicitations and deformations ?



$$\underline{x} = \underline{X} + \underline{\xi}(X, t)$$

Small deformation hypothesis

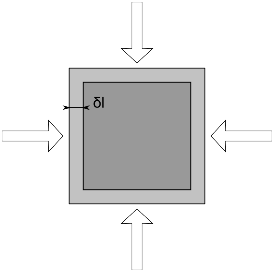
Most of the mechanics is made under this approximation. $|\partial \xi_i / \partial X_j| \ll 1$
Under this condition, we can define the strain tensor characterizing the change of length and angle of a material vector:

$$\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\nabla \underline{\xi} + \nabla^t \underline{\xi}\right) \qquad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i}\right)$$

Strains and strain tensor

Volumetric strain

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i}\right) \Rightarrow \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix}$$



$$\frac{d\Omega}{\Omega} = 3\frac{\delta l}{l} = 3\varepsilon = tr(\underline{\underline{\varepsilon}})$$

$$\underline{\underline{\varepsilon}} = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}$$

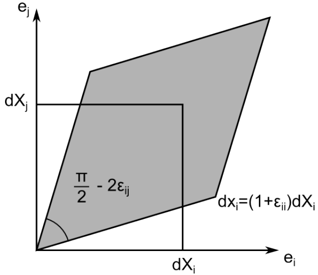
The diagonal term ε_{ii} of the linearized strain tensor corresponds to a volumetric deformation in the direction e_i.
The change in volume is obtained by $\Omega = (1 + \epsilon)\,\Omega_0$ with $\epsilon = tr(\underline{\underline{\varepsilon}})$

Strains and strain tensor

Deviatoric strain

The non diagonal terms are called *deviatoric*

These terms correspond to a change of orientation of a material vector.
The strain tensor is symmetric and its eigenvectors determine the *principal directions of deformation*



Interpretation of non diagonal terms

$2\varepsilon_{ij}$ is the change of angle undergone by two initially normal vectors in the directions i and j.

Decomposition into volumetric and deviatoric strains

$$\underline{\underline{\varepsilon}} = \frac{\epsilon}{3}\underline{\underline{1}} + \underline{\underline{e}}$$

Stresses and stress tensor

Body and surface forces

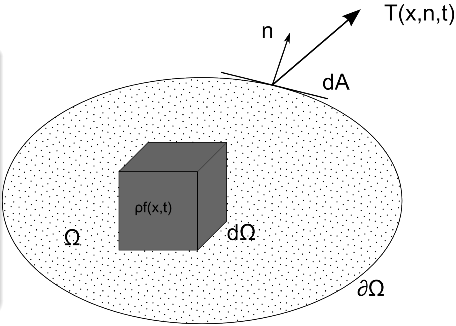
Body forces and contact forces

Body forces are applied on the whole domain Ω

- gravity
- electrostatic field

Contact forces are applied on the boundary of the domain $\partial\Omega$

- pressure
- shear

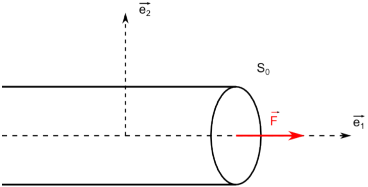


Static equilibrium of a domain

$\int_\Omega \rho \underline{f}(\underline{x}, t)\, d\Omega + \int_{\partial\Omega} \underline{T}(\underline{x}, t,)\, dA = 0$
n normal to the surface, ρ density of the solid

Stress vector

Stresses are contact forces

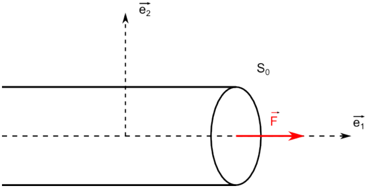


The expression of the stress vector depends on the direction we are looking at:

$\sigma_0 = \frac{F}{S_0} \qquad \tau_0 = 0$ and $\underline{T} = \sigma_0 \underline{e_1} + 0 \times \underline{e_2}$

Stress vector

Stresses are contact forces



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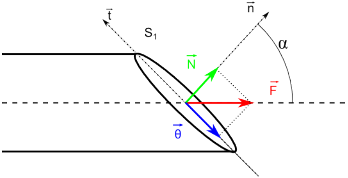


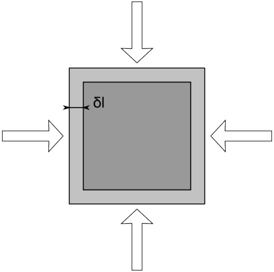
$\sigma_1 = \frac{F \cos \alpha}{S_1} \qquad \tau_1 = \frac{-F \sin \alpha}{S_1}$ and $\underline{T} = \sigma_1 \underline{n} + \tau_1 \underline{t}$

Stresses and stress tensor

The contact forces can be expressed as a result of a linear operator in a defined base

$\underline{T}\left(\underline{x}, t, \underline{n} = \underline{n}_j \underline{e}_j\right) = \underline{\underline{\sigma}} \cdot \underline{n} = \sigma_{ij} n_j \underline{e}_i$

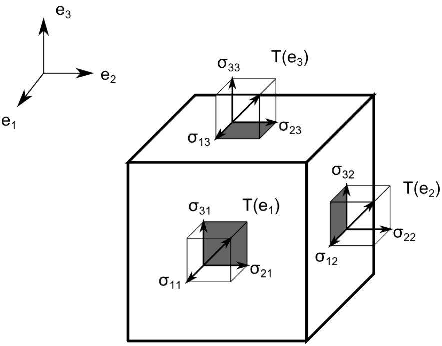
This defines the stress tensor



The stress here is analogous to a pressure. There is a negative sign because the normal to the face is toward the outside

$$\underline{\underline{\sigma}} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}$$

Stresses and stress tensor



Characteristics of the stress tensor

- symetric tensor. The eigenvector define the principal directions of stress
- σ_{ii} is the stress applying on the surface normal to $\underline{e}_i$ in the direction $\underline{e}_i$
  A pressure corresponds to a negative value
- σ_{ij} is the shear stress acting in along the $\underline{e}_i$ on the surface normal to $\underline{e}_j$
- the stress tensor can be decomposed as spherical part and deviatoric part
  $\underline{\underline{\sigma}} = \underline{\underline{s}} + \sigma \underline{\underline{1}}$ with $\sigma = 1/3 tr(\underline{\underline{\sigma}})$
- For a fluid, the shear stress is equal to zero and the spherical part reduces to the pressure with a $-$ sign

Terminology of states of stress

- Hydrostatic pressure : $\sigma_1 = \sigma_2 = \sigma_3$
  No shear stress on any plane $\rightarrow$ isotropic pressure
- Uniaxial stress :
  $\sigma_1 > \sigma_2 = \sigma_3 = 0$ (uniaxial traction)
  $\sigma_1 < \sigma_2 = \sigma_3 = 0$ (uniaxial compression)
- Axial stress with confinement : $\sigma_1 > \sigma_2 = \sigma_3 > 0$
- Pure triaxial stress : $\sigma_1 > \sigma_2 > \sigma_3$

Young relation

We can rewrite the Hooke's law with the tensor notation

In the basis of the principal directions of stress, we have the relation:

$$\begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} = \frac{\sigma}{E} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & -\nu \end{pmatrix}$$

In the general case, we have : $\varepsilon_{ij} = \frac{1}{E} \left[(1+\nu)\, \sigma_{ij} - \nu tr \left(\underline{\underline{\sigma}} \right) \delta_{ij} \right]$

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The mechanical equilibrium then writes $\int_\Omega \rho \underline{f} d\Omega + \int_{\partial\Omega} \underline{\underline{\sigma}} \cdot \underline{n} dA = 0$

Which reduces to $\nabla \cdot \underline{\underline{\sigma}} + \rho \underline{f} = 0$

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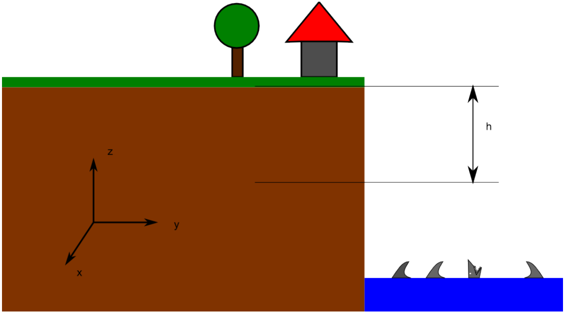
Strain work

The infinitesimal work dW supplied between t and $t + dt$ is:
$dW = \int_\Omega d\underline{\xi} \cdot \rho \underline{f} d\Omega + \int_{\partial\Omega} \left(d\underline{\xi} \cdot \underline{T}\right) dA$

Using the definition of the stress tensor we finally obtain the strain work for infinitesimal transformation:
$dW = \sum_i \sum_j \sigma_{ij} d\varepsilon_{ij}$ which also written as $dW = \underline{\underline{\sigma}} : d\underline{\underline{\varepsilon}}$

Landslides

Should you really build your house there ?



How can we assess the mechanical equilibrium of a soil ?

Mohr circles

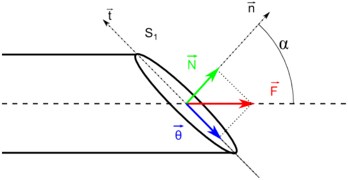
Mohr circles is a way to represent the stresses acting on the material point of a system subjected to a solicitation

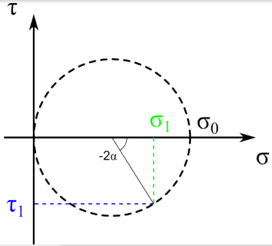
It is a 2D representation of the normal and shear stress with every possible orientation of the cutting plane

Mohr circle for a simple traction test

There is only one principal direction of stress: $\underline{\underline{\sigma}} = \begin{pmatrix} \sigma_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

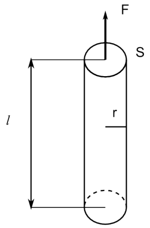
$$\begin{cases} \sigma_1(\alpha) = \frac{F \cos\alpha}{S_1(\alpha)} = \sigma_0 \cos^2\alpha \\ \tau_1(\alpha) = -\frac{F \sin\alpha}{S_1(\alpha)} = -\sigma_0 \cos\alpha \sin\alpha \end{cases} \Rightarrow \text{Parametric equation of a circle}$$

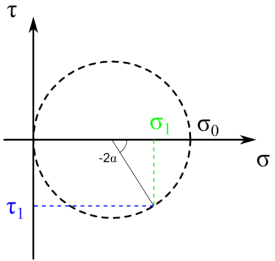




Mohr circles

Failure of a sample on a simple traction test



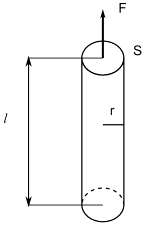


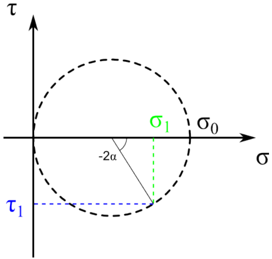
Failure of a ductile material is always due to the shear stress

The maximum shear stress is obtained for an orientation of the plane of 45° from the solicitation.
The failure will then follow this plane

Mohr circles

Failure of a sample on a simple traction test





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Mohr circles

More complex solicitations

Mohr circles for a biaxial solicitation $\underline{\underline{\sigma}} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

We choose a cutting plan with an angle α with $\underline{e_1}$ ($\underline{e_3}$ is the axis of rotation)
We can consider without any loss of generality that $\sigma_1 > \sigma_2$

The new normal and tangential components for this plan are:

$\sigma_n = \frac{F_1 \cos\alpha}{S_1} + \frac{F_2 \sin\alpha}{S_2} = \sigma_1 \cos^2\alpha + \sigma_2 \sin^2\alpha$

$\tau = -\frac{F_1 \sin\alpha}{S_1} + \frac{F_2 \cos\alpha}{S_2} = -\sigma \sin\alpha \cos\alpha + \sigma_2 \sin\alpha \cos\alpha$

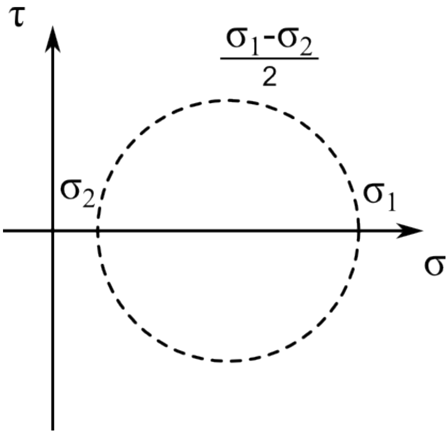
This corresponds to a circle with parametric equation:

$\sigma_n = \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2} \cos(-2\alpha)$

$\tau = \frac{\sigma_1-\sigma_2}{2} \sin(-2\alpha)$

Mohr circles

More complex solicitations

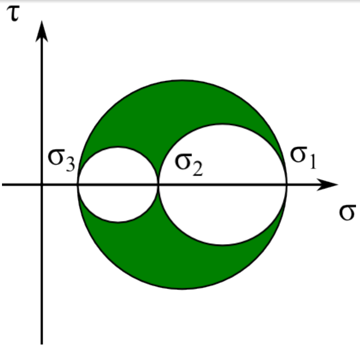


Mohr circles

More complex solicitations

Traxial solicitation: $\underline{\underline{\sigma}} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$

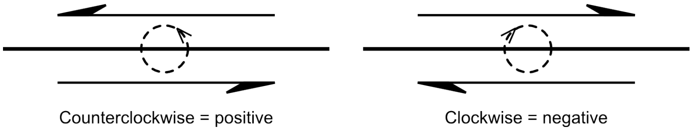
A triaxial solicitation can be considered as a superposition of 3 biaxial solicitations with the rotation of the plane around the 3 principal directions of stress. The result is a surface.



Sign convention in Mohr circles

Mohr circles are usually used in geotechnics: almost all stresses are compression

- compression stresses are positive (opposite from the normal case)
- The stress tensor is antisymmetric
  $\sigma_{12} = -\sigma_{21}$



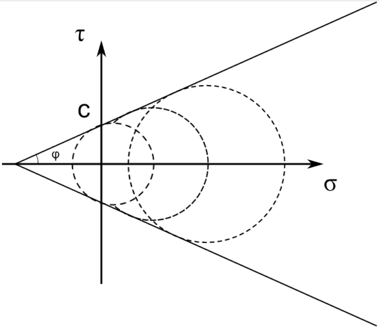
Failure of soils

Mohr-Coulomb criteron for compression failure

$\tau = \sigma \tan \varphi + c$

$\Rightarrow$ Failure of a granular material: shear of grains with respect to each other $\leftrightarrow$ friction angle φ

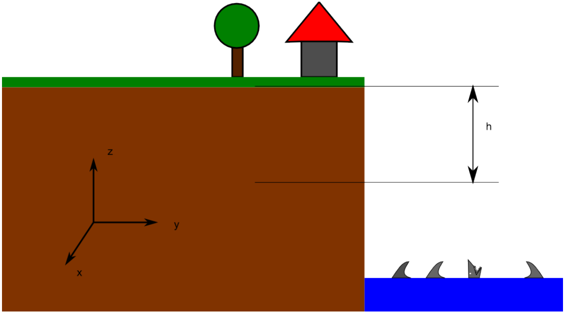
Criterion increasing with normal stress: consolidation



Failure occurs if a Mohr circle reaches the failure envelope. The failure plane orientation is given by the contact point between the circle and the envelope

Landslides

Should you really build your house there ?



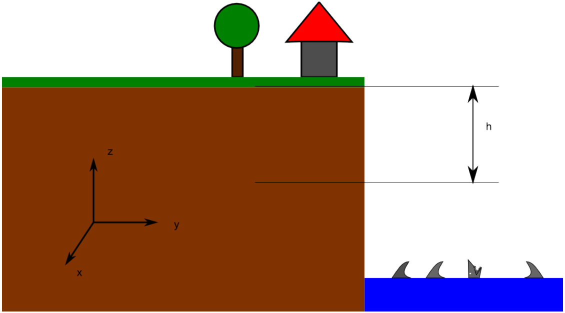
Stability of the soil without house

Vertical stress: $\sigma_{zz} = \rho g h$

Strain tensor: $\underline{\underline{\varepsilon}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix} \Rightarrow \begin{cases} 0 = \frac{1}{E}\left(\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right) \\ \varepsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right) \\ \varepsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right) \end{cases}$ Hooke's law

Landslides

Should you really build your house there ?



Static equilibrium

$$\nabla \cdot \underline{\underline{\sigma}} - \rho \underline{g} h = 0 \Rightarrow \begin{cases} \frac{\partial \sigma_{xx}}{\partial x} = 0 \\ \frac{\partial \sigma_{yy}}{\partial y} = 0 \\ \frac{\partial \sigma_{zz}}{\partial z} = 0 \end{cases}$$ At the cliff, $\sigma_{yy} = 0$ so $\forall y, \sigma_{yy} = 0$

Landslides

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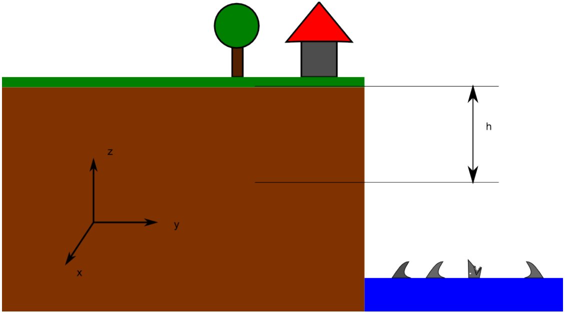
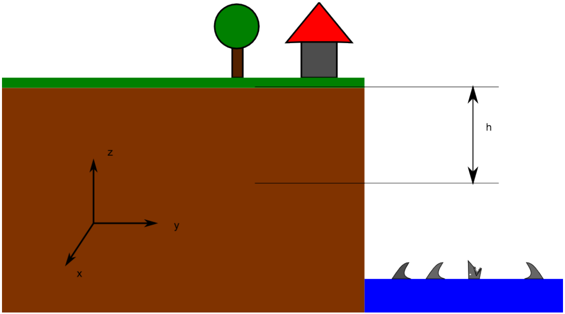


Resolution of the system of equations

$$\begin{cases} \sigma_{xx} = \nu \sigma_{zz} \\ \varepsilon_{yy} = \frac{\nu(1+\nu)}{R}\sigma_{zz} \\ \varepsilon_{zz} = \frac{1-\nu^2}{E}\sigma_{zz} \end{cases} \Rightarrow \underline{\underline{\sigma}} = \begin{pmatrix} \nu\rho g h & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho g h \end{pmatrix}$$

Landslides

Should you really build your house there ?

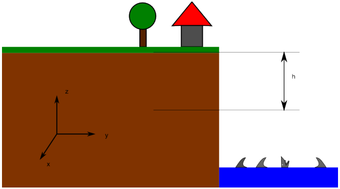


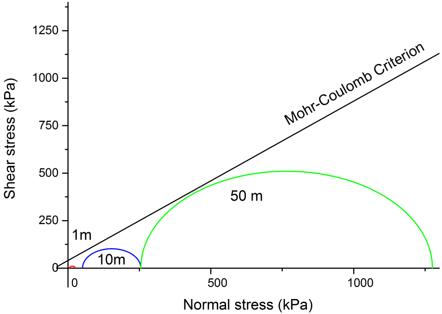
Stability of the soil

$$\begin{cases} \varphi = 40° \\ c = 40\text{kPa (loam)} \\ \nu = 0.2 \end{cases}$$

Landslides

Should you really build your house there ?

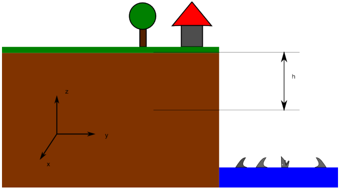


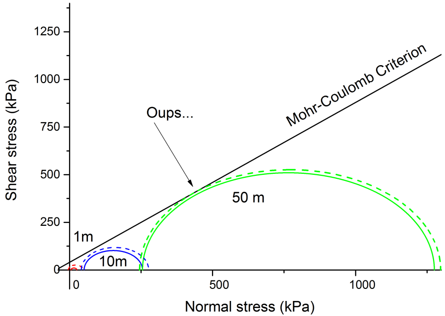


Influence of the house: $\sigma_{zz} = \rho g h + P$ (with $M_{house} = 40\text{kPa}$)

Landslides

Should you really build your house there ?

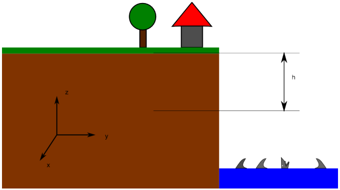


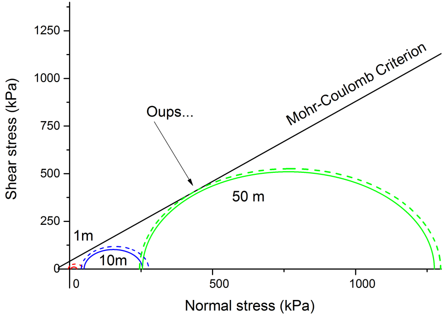


Angle of landslide ?

Landslides

Should you really build your house there ?





What of the influence of water in the interstitial space ?

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